Variations on the discrete Laplacian

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(Based on joint work with Adrien Poncelet)

Laplacian Δ is ubiquitous and connected with many different problems :

- (almost) every single solvable statistical models (Ising, dimers, free fermions)
- base for perturbation
- random walks, diffusive processes
- spanning trees, ...

As are spanning structures ...

- dimers (Temperley correspondence 1974)
- loop erased random walk (Lawler 1991, Pemantle 1991, Wilson 1996) and SLE_2 in the scaling limit (Lawler–Schramm–Werner 2004)
- sandpile model (Dhar–Majumdar 1992)

Great variety: spanning tree, spanning forests, unicycles, cycle-rooted spanning forests, cycle-rooted groves, incompressible spanning webs, ...

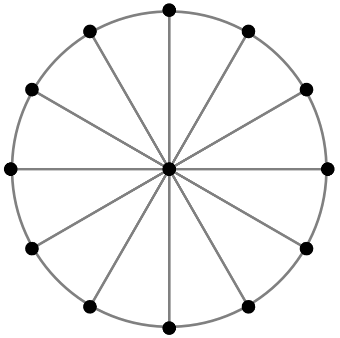
Spanning trees

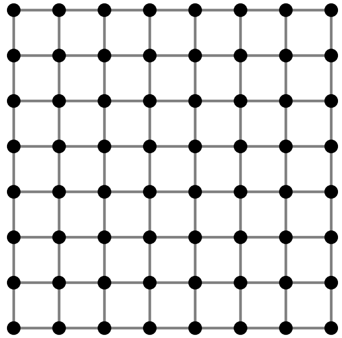
Most classical theme, in every textbook about graph theory or combinatorics.

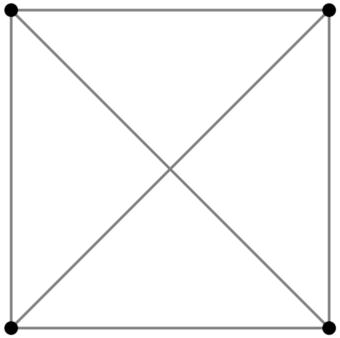
Basic setting = graph + set of vertices + set of edges between vertices.

(Here: single and unoriented edges)

Example of graphs







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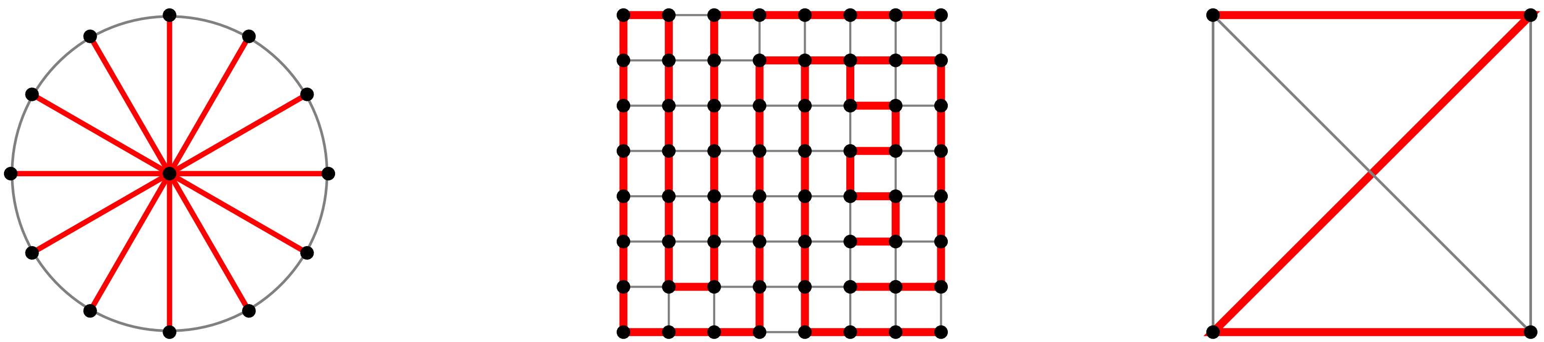
Example of spanning trees



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Spanning trees: counting

Most obvious question: how many spanning trees on a graph ?

First result attributed to Kirchhoff (1847), also known as matrix-tree theorem.

Form graph (or combinatorial) Laplacian matrix Δ,

$$\Delta_{u,v} = \begin{cases} \deg u = \#\,\text{incident edges} & \text{if } u = v, \\ -1 & \text{if } u, v \text{ are connected}, \\ 0 & \text{otherwise.} \end{cases}$$

Symmetric matrix, equal to minus adjacency matrix, up to diagonal.

Note all row and column sums vanish : Δ^{-1} does not exist !

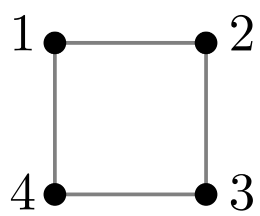
Kirchhoff's theorem : for a graph with N vertices,

$$Z = \text{number of spanning trees} = |\mathrm{ST}| = \frac{1}{N} \det{}' \Delta = \det \Delta^{\mathrm{rest}}$$

where Δ^{rest} is the restriction of Δ to graph with any vertex removed.

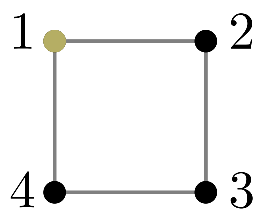
Z is a partition function w.r.t. uniform distribution on spanning trees (can be gener.)

Spanning trees: example of counting



$$\Delta = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \qquad \lambda = 0, 2, 2, 4, \quad |\mathrm{ST}| = 4$$

(unoriented)



1 removed $\rightarrow$ $\Delta^{\mathrm{rest}} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad \det \Delta^{\mathrm{rest}} = |\mathrm{ST}| = 4$

Note that Δ^{rest} is invertible !

Diagonal entries indicate that the edges pointing to 1 are kept $\longrightarrow$ **root**

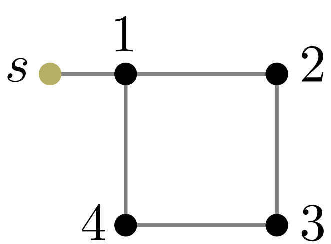
Rooted spanning trees

Removing vertex 1 (or any other) makes it special, and allows to define an orientation (most often conventional, except in sandpiles). All edges oriented towards the root :

(oriented)

Electrical image: seeing the edges as wires, connected at vertices, the root is the earth point, where current is flowing. Graph is said to be wired to the root.

Note that above is equivalent to connecting vertex 1 to extra vertex, taken as root s :



$$\Delta^{\mathrm{rest}} = \begin{pmatrix} 3 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}, \qquad \det \Delta^{\mathrm{rest}} = |\mathrm{ST}| = 4$$

Summary of trees

Spanning trees:

- loopless connected set of edges covering every vertex
- unique simple path connecting any pair of vertices (called chemical path)
- a spanning tree on N sites contains exactly $N-1$ edges
- cutting any single edge in a spanning tree makes it disconnected (a forest)
- (unoriented) spanning trees on graph $\Leftrightarrow$ rooted (oriented) spanning trees rooted at any particular vertex, reflected by $\frac{1}{N}\det'\Delta = \det\Delta^{\text{rest}}$
- root of a tree is unique (unique earth point !)
- rooting is very often a matter of convenience (unless you want to know if unique path from u to v goes through root, see later in sandpiles)

All very nice but ...

In many cases where spanning trees are central, connectivity properties are wanted $\longrightarrow$ need refined, richer structures: spanning forests !

Spanning forests

In short, spanning forests :

- subset of edges spanning all vertices, where each component is a tree
- requires to say which vertices are in which component
- rooted spanning forests : each component is oriented towards its own root

Following setting is sufficient in many situations.

Let $\mathcal{N}$ a set of $2k$ marked points, divided in $R = \{r_1, \ldots, r_k\}$ and $S = \{s_1, \ldots, s_k\}$.
Consider spanning forests with $k+1$ tree components :

- one tree component rooted at distinguished root s
- k tree components, each one containing a pair of marked vertices, one from R, the other from S
- pairing $\sigma \in \mathrm{S}_k$ says which pair $(r_i, s_{\sigma(i)})$ is on the same component

Quantity of interest : $Z\big[r_1, s_{\sigma(1)}\big| \ldots \big|r_k, s_{\sigma(k)}\big|s\big]$ = number of spanning forests with prescribed pairing of marked nodes

All-minors matrix-tree theorem

Gives relations between these partition functions.

Because the graph has distinguished root s, relevant Laplacian is Δ^{rest}, the graph Laplacian with s removed.

Let $G = (\Delta^{\text{rest}})^{-1}$, the Green function; set $G_R^S = (G_{r_i,s_j})_{1 \le i,j \le k}$.

All-minors matrix-tree theorem : for all divisions of $\mathcal{N}$ into $R \cup S$, we have

$$\det \Delta^{\text{rest}} \times \det G_R^S = \sum_{\sigma \in \mathrm{S}_k} \epsilon_\sigma \, Z\big[r_1, s_{\sigma(1)} \big| \ldots \big| r_k, s_{\sigma(k)} \big| s\big]$$

Remarks :

- yields system of relations since many choices of separating $\mathcal{N}$ into $R \cup S$
- concerns only pairings (two marked nodes on each tree), but no general result
- even then : the system is in general not invertible ! $\frac{1}{2}\binom{2k}{k}$ vs $(2k-1)!!$
  Solution : introduce a connection ...

Connection

Introduce a connection on edges: complex non-zero weights attached to oriented edges such $\phi_{u,v} = \phi_{v,u}^{-1}$.

Then define (line-bundle) Laplacian

$$\mathbf{\Delta}_{u,v} = \begin{cases} \deg u & \text{if } u = v, \\ -\phi_{u,v}^{-1} & \text{if } u, v \text{ are connected}, \\ 0 & \text{otherwise.} \end{cases}$$

Terminology refers to some sort of parallel transport

$$\mathbf{\Delta} f(u) = \sum_{\langle u,v \rangle} [f(u) - \phi_{u,v}^{-1} f(v)].$$

$\mathbf{\Delta}$ no longer symmetric, but invertible.

We will denote parallel transport on

- sequence of edge-connected vertices $(u_1, u_2, \ldots, u_\ell)$, by $\phi_{u_1 \to u_\ell} = \prod_i \phi_{u_i,u_{i+1}}$
- loop α of edge-connected vertices $(u_1, u_2, \ldots, u_1)$, by $\omega_\alpha = \prod_i \phi_{u_i,u_{i+1}}$.
  Called monodromy of α; it depends of the orientation of loop, $\omega_{\alpha^{-1}} = \omega_\alpha^{-1}$.

Rooted spanning groves

Combinatorial significance of non-trivial connection involves new structures:

> *a* ***cycle-rooted tree*** *(CRT) is a connected subset of edges containing a unique cycle (playing the role of root) with branches attached to it.*

A CRT therefore contains as many edges as sites.

To understand the combinatorial content of $\boldsymbol{\Delta}$, we need to combine components which are either trees or CRTs.

A rooted spanning grove is a subset of edges spanning all vertices, such that each component is either

- a tree rooted at a root s,
- a tree containing a pair of marked vertices
- a cycle-rooted tree containing no marked vertices.

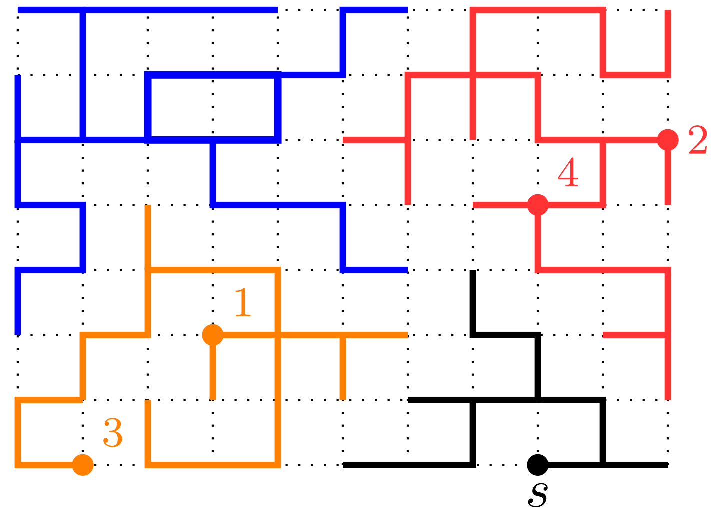
The first two types of trees form a spanning forest, in the previous sense; third one (CRTs) are new and are a consequence of non-trivial connection.

Rooted spanning groves (RSG)

Let $\mathcal{N}$ a set of $2k$ marked points, $R = \{r_1, \ldots, r_k\}$ and $S = \{s_1, \ldots, s_k\}$.

Denote by σ a pairing of marked points, s.t. $(r_i, s_{\sigma(i)})$ are on the same tree component. Each pair has parallel transport $\phi_{s_{\sigma(i)} \to r_i}$.

Weigh a CRT component with cycle α by factor $2 - \omega_\alpha - \omega_\alpha^{-1}$.



Define the partition function of σ-RSG by

$$\mathscr{Z}\left[\begin{smallmatrix} s_{\sigma(1)} \\ r_1 \end{smallmatrix} \middle| \ldots \middle| \begin{smallmatrix} s_{\sigma(k)} \\ r_k \end{smallmatrix} \middle| s \right] = \sum_{\sigma-\mathrm{RSG}} \prod_{\alpha \in \sigma-\mathrm{RSG}} (2 - \omega_\alpha - \omega_\alpha^{-1}) \times \prod_i \phi_{s_{\sigma(i)} \to r_i}$$

When connection gets trivial, $\phi \to 1$, reduces to number of spanning forests with prescribed marked points.

As the graph has distinguished root s, denote by $\mathbf{\Delta}^{\mathrm{rest}}$ the Laplacian with connection, where row and column s are removed; let $\mathbf{G} = (\mathbf{\Delta}^{\mathrm{rest}})^{-1}$.

Rooted spanning groves

Following result is the generalization of spanning forest theorem (all-minors matrix-tree) for Laplacian with connection.

Grove theorem (Kenyon-Wilson, 2011 publ. 2015) : for all divisions of $\mathcal{N}$ as $R \cup S$,

$$\det \mathbf{\Delta}^{\mathrm{rest}} \times \det \mathbf{G}_R^S = \sum_{\sigma \in \mathrm{S}_k} \epsilon_\sigma \, \mathscr{Z}\left[\begin{smallmatrix} s_{\sigma(1)} \\ r_1 \end{smallmatrix} \middle| \ldots \middle| \begin{smallmatrix} s_{\sigma(k)} \\ r_k \end{smallmatrix} \middle| s \right]$$

As before, $\mathbf{G}_R^S = (\mathbf{G}_{r_i, s_j})_{i,j}$ is $k \times k$ restriction of $\mathbf{G}$ (as before).

Remarks:

- if trivial connection, reduces to all-minors tree-matrix theorem
- unlike the all-minors tree-matrix theorem, yields invertible linear system for $\mathscr{Z}\left[\begin{smallmatrix} s_{\sigma(1)} \\ r_1 \end{smallmatrix} \middle| \ldots \middle| \begin{smallmatrix} s_{\sigma(k)} \\ r_k \end{smallmatrix} \middle| s \right]$ (at least in certain cases, depends on topology).
- There is a price to pay ...

Loop-erased random walk

LERW (introduced by Lawler, 1980) :

starting from x_1, run a symmetric random walk RW on $\mathbb{Z}^2$, and erase loops in chronological order, as they arise.

Result is a simple path γ, by definition, a sample of LERW.

Measure on LERW paths $\gamma : x_1 \to x_2$:

$$\mu(\gamma_{x_1,x_2}) = \sum_{\mathrm{RW}\downarrow\gamma_{x_1,x_2}} \mu(\mathrm{RW}), \qquad \text{with } \mu(\mathrm{RW}) = 4^{-|\mathrm{RW}|}$$

LERW is **different** from symmetric RW and SAW

- fractal dimension of LERW paths γ is $\frac{5}{4}$ (is $\frac{3}{2}$ for Brownian curves, $\frac{4}{3}$ for SAW)
  It takes about $N^{5/4}$ edges to get to distance N (vs N^2 edges for RW)
- described by SLE$_{\kappa=2}$ in continuum limit ($\kappa = 4$ for BM, $\kappa = \frac{8}{3}$ for SAW)

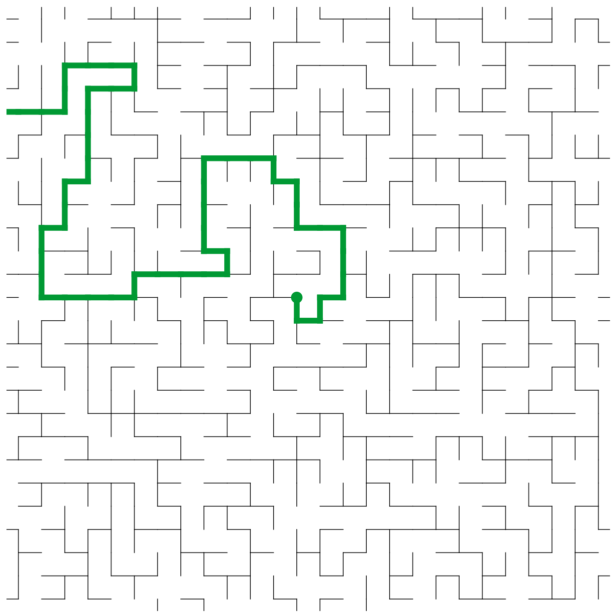
LERW and spanning trees

Well-known correspondence.

On random spanning trees, unique random simple path γ between x_1 and x_2, a chemical path.

If spanning trees are sampled according to uniform distribution, the measure induced on chemical paths $x_1 \to x_2$ is identical to measure on LERW paths from x_1 to x_2 !

(Pemantle, 1991)



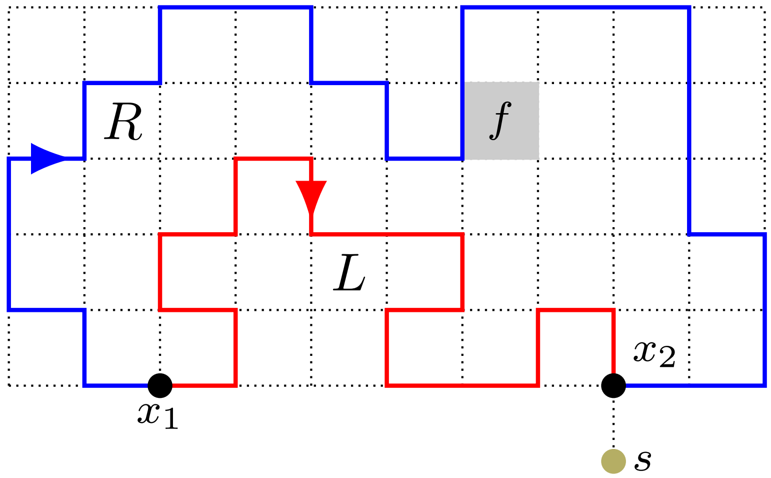
(Figure from Kenyon-Wilson)

Take x_1, x_2 on boundary of rectangular graph, and fix a face f in bulk.

Question : what is the probability that unique path $\gamma : x_1 \to x_2$ leaves f to its left ?

(discrete analogue of Schramm's formula)

Groves : application 1



Rectangular graph with x_1, x_2 on lower boundary; root is only connected to x_2.

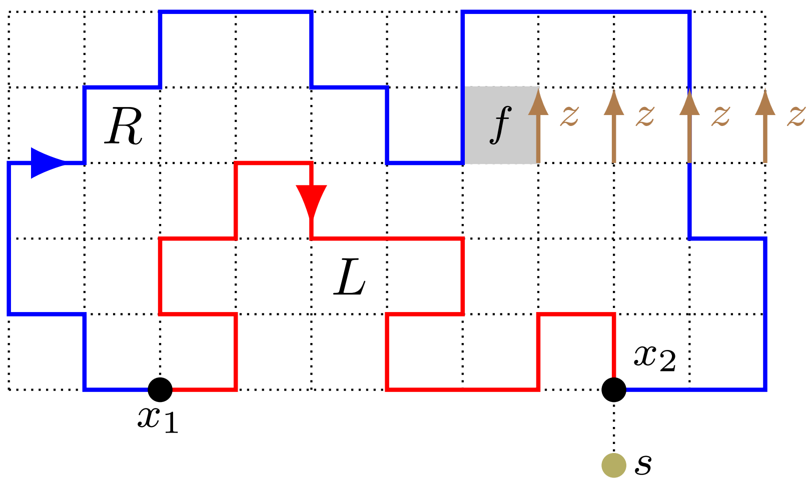
Take $\mathcal{N} = \{x_1, x_2\}$ as marked vertices.

Interested in spanning forests $Z[1, 2|s]$ where $1, 2$ are on one tree, and s on another tree (on its own then).

Know for sure that x_1 is path-connected to x_2, but no control of where the path goes.

Solution : introduce a constant, non trivial connection between f and boundary !

Groves : application 1



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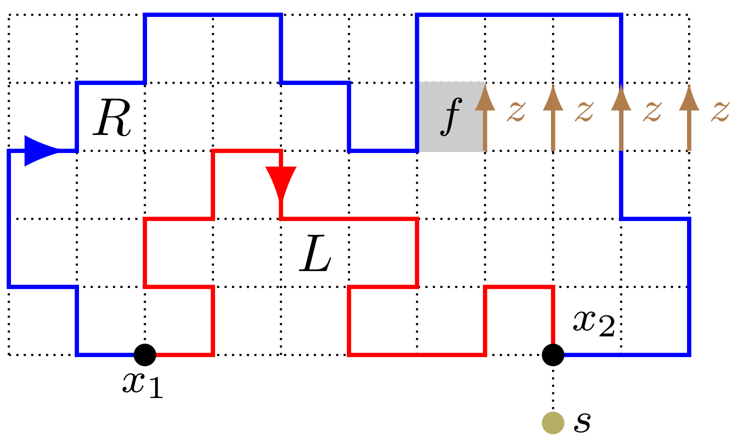
Arrow $u \xrightarrow{z} v$ means connection $\phi_{u,v} = z$ and $\phi_{v,u} = z^{-1}$.
What changes ?

- now talk of spanning groves $\mathscr{Z}\left[\begin{smallmatrix}1\\2\end{smallmatrix}|s\right]$ or $\mathscr{Z}\left[\begin{smallmatrix}2\\1\end{smallmatrix}|s\right]$, namely two trees **and** CRTs looping around f (only them have nontrivial monodromy $\omega_\alpha = z$ or z^{-1})
- allows to disentangle the two types of path, L and R !

Groves : application 1

Remember weight of grove contributing to $\mathscr{Z}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\middle|s\right]$ or $\mathscr{Z}\left[\begin{smallmatrix}2\\1\end{smallmatrix}\middle|s\right]$ is resp.

$$\prod_{\alpha\in \mathrm{RSG}} (2-\omega_\alpha-\omega_\alpha^{-1}) \times \Big(\phi_{1\to 2} \ \text{ or } \ \phi_{2\to 1}\Big)$$



Compute the parallel transports :

$\phi_{1\to 2} = 1$ for L $\qquad$ $\phi_{1\to 2} = z^{-1}$ for R

$\phi_{2\to 1} = 1$ for L $\qquad$ $\phi_{2\to 1} = z$ for R

We decompose

$$\mathscr{Z}\left[\begin{smallmatrix}1\\2\end{smallmatrix}\middle|s\right] = \mathscr{Z}_L\left[\begin{smallmatrix}1\\2\end{smallmatrix}\middle|s\right] + \mathscr{Z}_R\left[\begin{smallmatrix}1\\2\end{smallmatrix}\middle|s\right] = p_L(z) + z^{-1}\, p_R(z)$$

$$\mathscr{Z}\left[\begin{smallmatrix}2\\1\end{smallmatrix}\middle|s\right] = \mathscr{Z}_L\left[\begin{smallmatrix}2\\1\end{smallmatrix}\middle|s\right] + \mathscr{Z}_R\left[\begin{smallmatrix}2\\1\end{smallmatrix}\middle|s\right] = p_L(z) + z\, p_R(z)$$

with $p_{L,R}(z)$ is the contribution of the groves s.t. the path $1\to 2$ leaves f to its L,R.

Groves : application 1

Then

$$\det \boldsymbol{\Delta}^{\text{rest}} \times \mathbf{G}_{2,1} = p_L(z) + z^{-1}\, p_R(z)$$
$$\det \boldsymbol{\Delta}^{\text{rest}} \times \mathbf{G}_{1,2} = p_L(z) + z\, p_R(z)$$

yields

$$p_L(z) = \frac{\det \boldsymbol{\Delta}^{\text{rest}}(\mathbf{G}_{2,1} - z^{-2}\mathbf{G}_{1,2})}{1 - z^{-2}}, \qquad p_R(z) = \frac{\det \boldsymbol{\Delta}^{\text{rest}}(\mathbf{G}_{1,2} - \mathbf{G}_{2,1})}{z(1 - z^{-2})}$$

and also

$$\frac{p_L(z)}{p_L(z) + p_R(z)} = \frac{\mathbf{G}_{2,1} - z^{-2}\mathbf{G}_{1,2}}{(z-1)(\mathbf{G}_{2,1} + z^{-1}\mathbf{G}_{1,2})}.$$

The probability we are after is the limit $z \to 1$ of last ratio

$$\mathbb{P}(\gamma_{x_1,x_2} \in L) = \lim_{z \to 1} \frac{\mathbf{G}_{2,1} - z^{-2}\mathbf{G}_{1,2}}{(z-1)(\mathbf{G}_{2,1} + z^{-1}\mathbf{G}_{1,2})}$$

since $p_L(1)$ is fraction of spanning trees s.t. $\gamma_{x_1,x_2} \in L$ and all cycles disappear.

Groves : application 1

To analyze

$$\mathbb{P}(\gamma_{x_1,x_2} \in L) = \lim_{z \to 1} \frac{\mathbf{G}_{2,1} - z^{-2}\mathbf{G}_{1,2}}{(z-1)(\mathbf{G}_{2,1} + z^{-1}\mathbf{G}_{1,2})}$$

remember that $\mathbf{G} = (\boldsymbol{\Delta}^{\text{rest}})^{-1}$.

For $z = 1$, Δ^{rest} is symmetric $\Rightarrow G$ is symmetric too, so that both numerator and denominator vanish when $z \to 1$.

Expanding $\mathbf{G}$ around $z = 1$,

$$\mathbf{G}_{u,v} \equiv G_{u,v} + (z-1)G'_{u,v} + \ldots$$

L'Hospital's rule yields, using $G_{u,v} = G_{v,u}$ and $G'_{u,v} = -G'_{v,u}$ (see next)

$$\mathbb{P}(\gamma_{x_1,x_2} \in L) = 1 - \frac{G'_{1,2}}{G_{1,2}}$$

(Kenyon 2011)

Check A. Poncelet (2017, to appear) for concrete formulas in different geometries.

Derivative of Green function

Write $\mathbf{\Delta}(z) = \Delta + B$, where $B_{u,v} = 1 - \phi_{u,v}^{-1}$ for nodes $\langle u, v \rangle$ (n.n.).

Then

$$\mathbf{G} = (\Delta + B)^{-1} = \left[\Delta(\mathbb{I} - \Delta^{-1} B)\right]^{-1} = (\mathbb{I} - \Delta^{-1} B)^{-1} \Delta^{-1}$$
$$= (\mathbb{I} - GB)^{-1} G = (\mathbb{I} + GB + \ldots) G$$

Expanding in powers of $1 - z$ at first order, we get

$$\mathbf{G}_{u,v} = G_{u,v} - (1 - z) \sum_{k,l \,:\, \phi_{k,l} = z} \left[G_{u,k} G_{l,v} - G_{u,l} G_{k,v}\right] + \ldots,$$

and

$$G'_{u,v} = - \sum_{k,l \,:\, \phi_{k,l} = z} \left[G_{u,k} G_{l,v} - G_{u,l} G_{k,v}\right] = -G'_{v,u}.$$

Depends explicitly on the location of non-trivial connection.

For u close to v, can be reduced to a finite sum.

Abelian sandpile model

Take a grid Λ with N sites.

Attach a random variable $h_i = 1, 2, 3, 4$ to every site (h_i is # grains).

2	3	1	3	4	2	1	4	2	3
4	2	3	1	3	2	4	1	2	1
2	2	1	1	4	3	4	2	3	2
2	2	1	2	4	2	1	3	2	3
3	4	3	2	1	1	3	4	3	4
4	4	3	2	4	3	2	1	2	3
2	3	3	4	4	3	1	1	2	3
2	3	2	4	3	3	4	2	4	3
3	1	3	2	4	2	1	4	4	3
4	3	2	4	3	1	2	3	4	1

stable configs $= 4^N$

Need a probability measure on configurations : induced by dynamics

ASM is dynamical

ASM is a dynamical system : $\mathcal{C}_t \xrightarrow{\mathcal{T}} \mathcal{C}_{t+1}$.

Defined in two steps :

1. on **random** site i of $\mathcal{C}_t$, drop one grain: $h_i \to h_i + 1$
2. relaxation: all unstable sites topple (avalanche)

If $h_i \geq 5$, then $\begin{cases} h_i \to h_i - 4 \\ h_j \to h_j + 1, \quad j = \text{nearest neighbour of } i \end{cases}$

Repeat until all sites are stable again $\longleftarrow$ OK BECAUSE DISSIPATION !!

Resulting configuration is $\mathcal{C}_{t+1}$. (on boundaries)

Potential chain reaction: one grain dropped can trigger a large avalanche.

Avalanches spanning whole system will happen, and induce correlations of heights over long distances $\longrightarrow$ as dynamics runs, system will enter a critical state.

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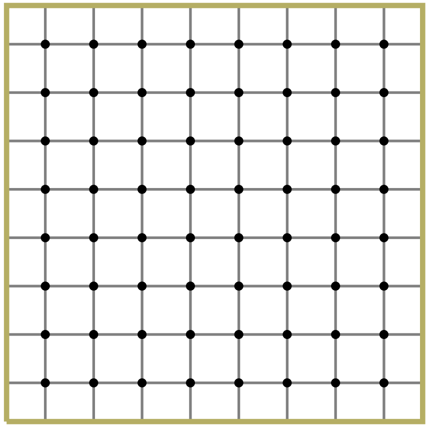
   Resulting configuration is $\mathcal{C}_{t+1}$. (on boundaries)

Upon toppling, redistribution of sand described by toppling matrix according to $h_j \to h_j - \Delta^{\text{rest}}_{j,i}$ for all j, where

Δ^{rest} is restriction of the graph Laplacian of $\longrightarrow$

Root is a sink that collects sand falling off.

Sink is special : receives sand, but never topples itself.



Recurrent configurations

- Dynamics is stochastic $\rightarrow$ proba distribution $\mathbb{P}_t(\mathcal{C})$ on set of configs.
- Certain configs, called transient, have a zero probability to occur after the dynamics has been run for long enough.
  The image of the repeated dynamics $\mathcal{T}$ shrinks and then stabilizes.
- Unique probability measure $\mathbb{P}$ invariant under dynamics

$$\mathbb{P}(\mathcal{C}) = \lim_{t\rightarrow\infty} \mathcal{T}^t\, \mathbb{P}_0(\mathcal{C})$$

- $\mathbb{P}$ is uniform on recurrent configs, which keep reoccurring under the dynamics, all with equal probability.
- Recurrent configs are 1-to-1 correspondence with spanning trees, rooted at sink.
  Explicit, non-local mapping heights $\leftrightarrow$ spanning trees, by **burning algorithm**.
  Sandpile model with measure $\mathbb{P}$ is equivalent to uniform spanning tree problem.
- By Kirchhoff's theorem, number of recurrent configs (partition function) is

$$Z \equiv \#\ \text{spanning trees} = \det \Delta^{\text{rest}} \simeq 3.21^N \ll 4^N$$

- Recurrent configurations are subjected to non-local constraints.

ASM summary

1. ASM = recurrent configuration space + measure $\mathbb{P}$.
2. At finite volume, $\mathbb{P}$ is unique but explicitly depends on type of lattice, size of lattice, boundary conditions, number and location of dissipative sites, ...
3. Measure allows to compute joint probabilities, typically $\mathbb{P}[h_i = a, h_j = b, \ldots]$.
4. Want to study these in large/infinite volume and relate to conformal field predictions, on the basis that

   scaling $\lim_{|\Lambda|\to\infty} \mathbb{P}_\Lambda$ is field-theoretic measure of a LogCFT

   F.i. known (1st and 2nd) and conjectured (3rd) that

$$\mathbb{P}_c[h_i = 1, h_j = 1] \sim \frac{1}{|i-j|^4}, \quad \mathbb{P}_c[h_i = 1, h_j > 1] \sim \frac{\log|i-j|}{|i-j|^4}, \quad \mathbb{P}_c[h_i > 1, h_j > 1] \sim \frac{\log^2|i-j|}{|i-j|^4}$$

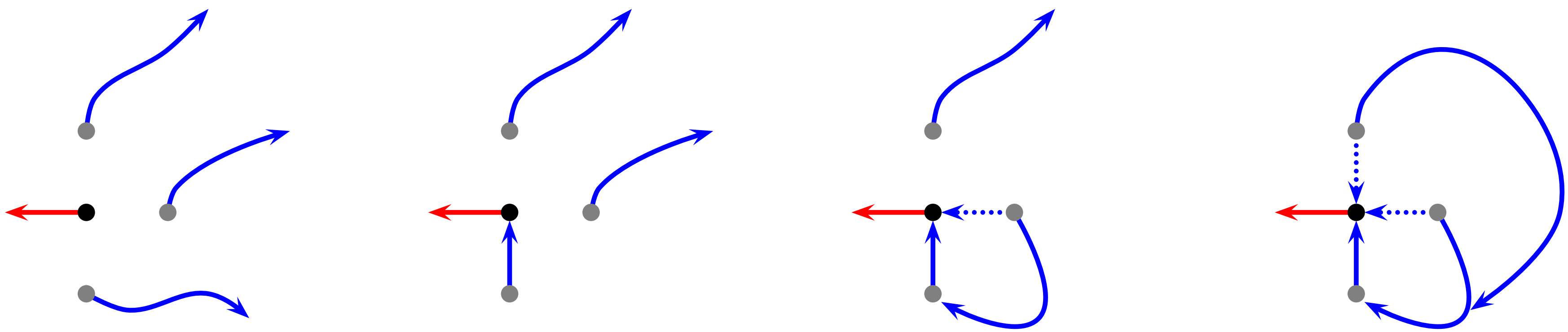
5. However non-local nature of recurrent configs/spanning trees makes life hard ...
6. Here, mere illustration of difficulties and relevance of groves !

1-site probabilities : trees, branches, leaves

Want to compute $\mathbb{P}[h_i = a]$ for $a = 1, 2, 3, 4$.

Need to translate heights in terms of characteristics of spanning trees.

Turns out (burning algor.) that h_i is determined by number of **predecessors** of i among its nearest neighbours



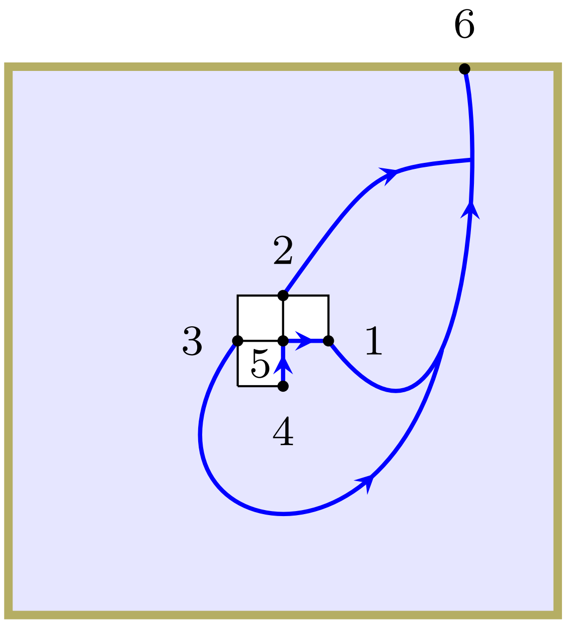
height ≥ 1 height ≥ 2 height ≥ 3 height ≥ 4

Height ≥ 2 : have to control long paths across whole lattice $\longrightarrow$ non-local !!

Height 1 is easy, heights 2, 3, 4 much harder !! Took 15 years to find closed form ...

Spanning trees with 1 predecessor

Want to compute X_1, fraction of spanning trees for which site i has 1 predecessor.
In infinite volume limit ($\mathbb{Z}^2$), which neighbour is predecessor is irrelevant, say South.
Outgoing arrow from i also irrelevant, say East. Multiply result by $4 \cdot 3 = 12$.



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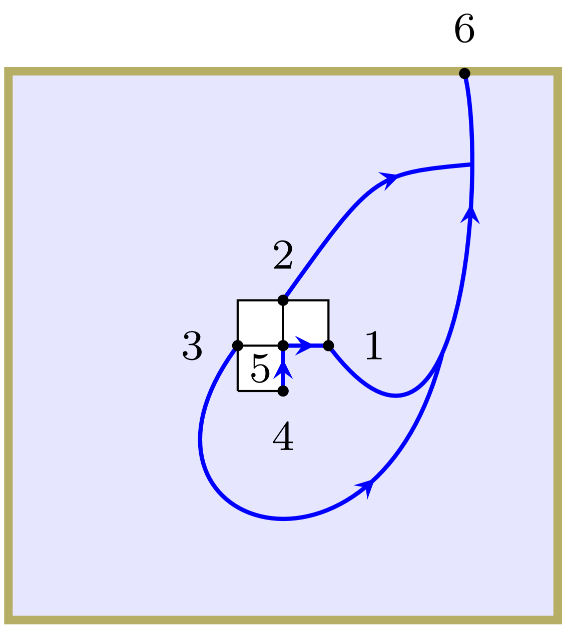

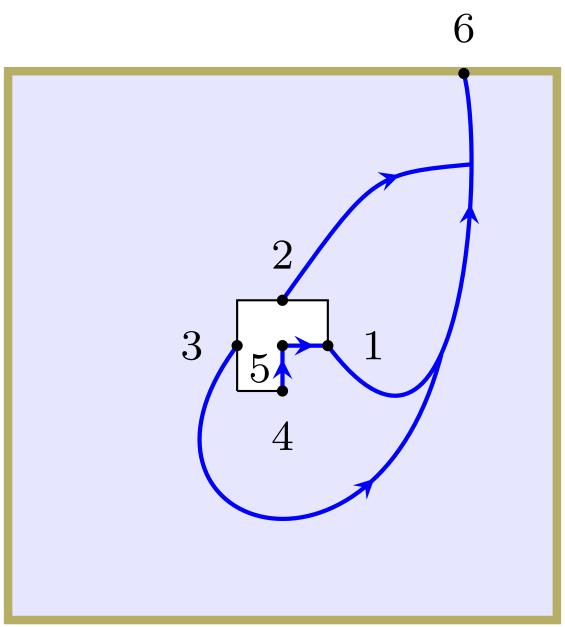

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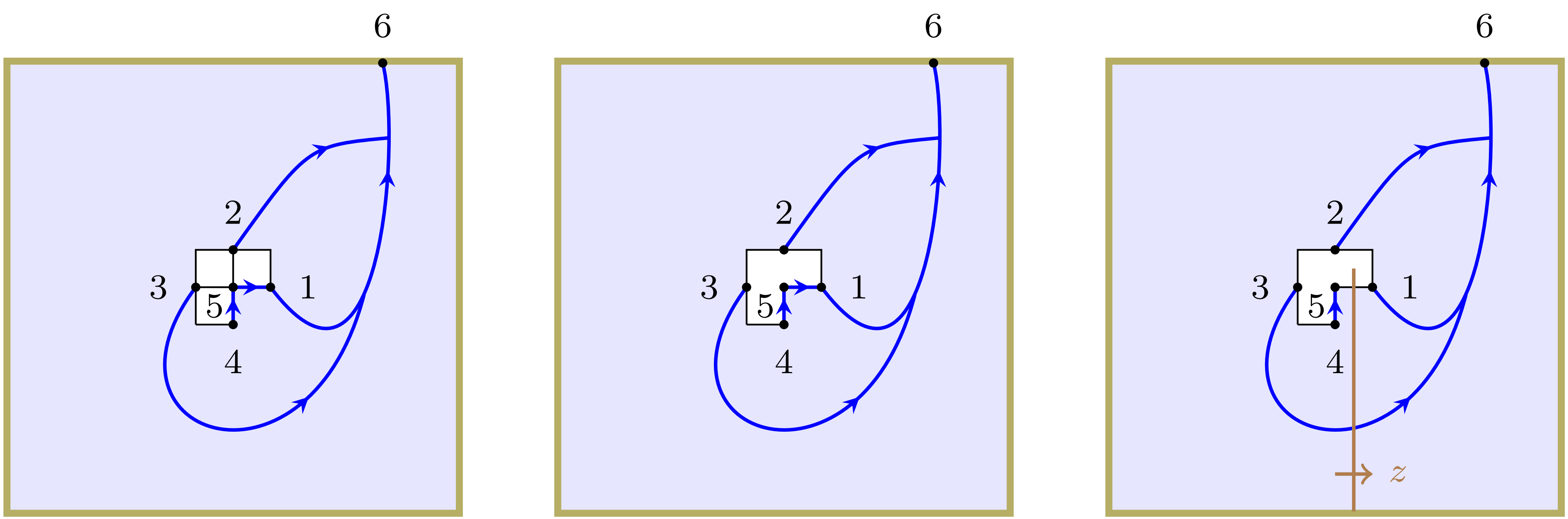

$$X_1 = 12\,\frac{\overline{Z}[45|1236]}{Z}, \qquad \overline{Z}[45|1236] = \overline{Z}[45|136] = \overline{Z}[45|16] - \overline{Z}[345|16] = \overline{Z}[45|16] - \overline{Z}[35|16]$$

Reduces to counting groves with pairs of nodes. Insert connection and compute !

Spanning trees with 1 predecessor

Insert connection, compute grove fractions and take limit of trivial connection.

Result reads

$$X_1 = 12 \cdot \frac{\overline{Z}}{Z} \left[\overline{G}_{4,4} - \overline{G}_{4,5} - \overline{G}_{3,4} + \overline{G}_{3,5} + \overline{G}'_{3,4} - \overline{G}'_{3,5} - \overline{G}'_{4,5} \right].$$

Remains to express Green function on modified lattice in terms usual Green function.

Final result reads

$$\boxed{X_1 = \frac{3}{4} - \frac{3}{2\pi} - \frac{15}{\pi^2} + \frac{48}{\pi^3} \simeq 0.300\,791.}$$

Details omitted, a matter of a few hours calculations. All reduces to finite number of usual Green functions, for close points.

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